Physics 15c; REV 0[1]
January 31, 2007

The Pendulum

Laboratory 1, Physics 15c

Due Friday, February 16, in front of Sci Cen 301

1 Introduction

Most oscillating systems behave like simple harmonic oscillators for sufficiently small displacements. This approximation can be applied to topics in many areas of physics. It is interesting to examine how these systems behave under conditions where the approximation is measurably inaccurate.

When making approximations for small oscillations, one usually looks at the Taylor series expansion of some function, and then ignores all but the leading one or two terms. In this lab, we will examine the validity of terminating a Taylor series after the first or second term as we use these expansions to simplify the behavior of some oscillating systems.

Another element to the experiments is to work with error analysis. When performing a calculation using quantities that have been measured, it is necessary to include the uncertainties associated with the measurements of these quantities in the calculation. When this is done correctly, any final calculated value should have an associated uncertainty that includes in its range the *accepted* value. We will apply this error analysis to a precise measurement of the gravitational acceleration, g.

2 Simple Pendulum

2.1 Objective of this experiment

Theory tells us that the behavior of a simple pendulum can be approximated by that of a simple harmonic oscillator for small amplitudes. Our goal is to determine for what initial displacement θ_0 do we begin to measure a noticeable deviation from the approximation used in the theory.

2.2 Theory

Recall that many functions $f(x)$ can be expanded in a Taylor series as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

So $\sin x$ can be expanded as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

[1]Anita's revisions added to Andrey's; δq /q changed to δq, end of p. 6, 9/28/04.

Now consider the dynamics of a simple pendulum, in Figure 1, below:

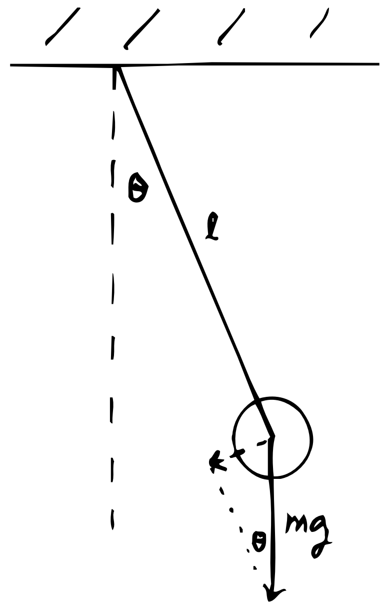

Figure 1: Forces on a simple pendulum

For rotational motion, $\tau = I\alpha$, where $\alpha = \ddot{\theta}$ (angular acceleration), so we have

$$-(mg\sin\theta)l = I\ddot{\theta} = ml^2\ddot{\theta},$$
$$-g\sin\theta = l\ddot{\theta},$$
$$\ddot{\theta} + \tfrac{g}{l}\sin\theta = 0.$$

For *small* displacements θ, we can approximate $\sin\theta$ by θ, as can be seen from the above expansion of $\sin x$ (since for small x, $x^3 \ll x$). In this approximation our equation of motion becomes

$$\ddot{\theta} + \tfrac{g}{l}\theta = 0.$$

This is similar to the equation of motion for an undamped simple harmonic oscillator:

$$\ddot{x} + \tfrac{k}{m}x = 0.$$

We know that for a SHO, the period of oscillation is given by

$$T = 2\pi\sqrt{\tfrac{m}{k}};$$

therefore we expect the period of a simple pendulum to be $T = 2\pi\sqrt{\frac{l}{g}}$. The period T is independent of the amplitude of oscillations for sufficiently small initial displacements.

2.3 Experiment 1: angle at which easy approximation breaks down

The preceding discussion should give us an idea for finding the angular displacement at which a simple pendulum no longer behaves like a SHO, or in other words, the angle at which the approximation $\sin\theta \approx \theta$ breaks down. If we measure the period of the pendulum as a function of initial displacement θ_0, we expect to measure a period which is constant for a range of angles, from 0 to θ_c. We call θ_c the angle at which we definitively measure a deviation from constant period. According to our theory, there is always a finite x^3 contribution (for finite x). Thus, the more accurate the measurement, the smaller θ_c becomes.

Procedure

Read through all of this procedure section before beginning your measurements. The notes at the end of the procedure section are especially relevant.

In this section of the lab you will measure the time it takes for the pendulum to oscillate through five cycles (*i.e.* measure $5T$) for different initial displacements.

1. First, make 10 measurements of $5T$ at a fixed 5-degree angle, to measure the spread of your measurements. Make a histogram of all of these $5T$ measurements. Verify that the data can be approximated to a Gaussian distribution. The spread of the distribution should be a reasonably good approximation of the uncertainty σ_{5T} of one single measurement of $5T$.

2. Then measure $5T$ N times at various angles (N will be given by your lab TF). You should make measurements at 5 degrees and then continue at 5 degree increments. At each angle, find the average of $5T$ ($\overline{5T}$) and the standard-deviation of this average δ_{5T}. Note, that the standard-deviation of this 5T measurement is the standard-deviation of a single measurement of $5T$: σ_{5T}. The standard-deviation of the average (mean) is the standard deviation of a single measurement divided by $\sqrt{N}$ where N is the number of measurements according to the central limit theorem. That is $\delta\, 5T = \frac{\sigma_{5T}}{\sqrt{N}}$. Calculate these values and plot them on a graph before you move on to measure the next angle.

3. Find the angle θ_c at which you definitively measure a period that deviates from its (initially) constant value. The deviation should be small, but larger than your experimental error, so that you know it is a real deviation. This occurs when the error bar in the average of 5T does not overlap the average 5T range that you found for small θ.

4. Make measurements for angles 15 degrees past this point.

5. Plot $5T$ versus θ_0 with the appropriate error bars on both the average-of-5T at small θ (5 degrees), and the average-of-5T at each θ increment.

Note:

- You should think ahead of time what is the most effective way for you and your partner(s) to take your readings. For example, your measurements will be more accurate if you start timing after a half cycle instead of when you release the pendulum bob.

- Design your measurement to minimize sources of systematic error like friction.

- Understand that this (like all real measurements) has finite accuracy. Your value for θ_c will not be accurate to more than several degrees.

Conclusions and Questions

For your value of θ_c, calculate the ratio of the second term to the first term in the Taylor series expansion of $\sin\theta$. Do you think it is of any value to approximate this system by a SHO?

Think about the sources of error in your measurement of $5T$. How could you reduce some of these? Is it better to measure five periods at one time (as you did) or one period five times? Why? Why did we measure five oscillations at one time, and not more or less?

3 Experiment 2–Precise Measurement of g

3.1 Objective

Our goal is to make a precise measurement of the gravitational acceleration g using an electronic timing circuit. Using proper uncertainty considerations should enable us to determine a range of values for g. We hope that the established value (what does that mean?) will fall in that range.

3.2 Theory-Kater's Pendulum

A physical pendulum has its mass distributed along its entire length, whereas a simple pendulum has its mass concentrated at the end of a "massless" string. The period of a physical pendulum depends on the moment of inertia of the device which, in practice, is almost impossible to determine accurately. One can eliminate the dependence of the period of a physical pendulum on I by using a Kater's pendulum, which is a physical pendulum that has a pivot point (a knife edge) on each end. If the length L between these pivot points and the mass distribution of the pendulum are adjusted appropriately, it is possible to obtain identical periods about each pivot. In this case, the moment of inertia can be eliminated from the expression for the period of the pendulum, and the period can be written simply as $T = 2\pi\sqrt{\frac{L}{g}}$where L is the distance between the two pivot points. Our goal for this experiment will be to obtain measurements of T and L, which will allow us to determine g.

3.3 Theory-Electronics

Below (Figure 2) is the schematic for a timing circuit that we have built for you on a breadboard.

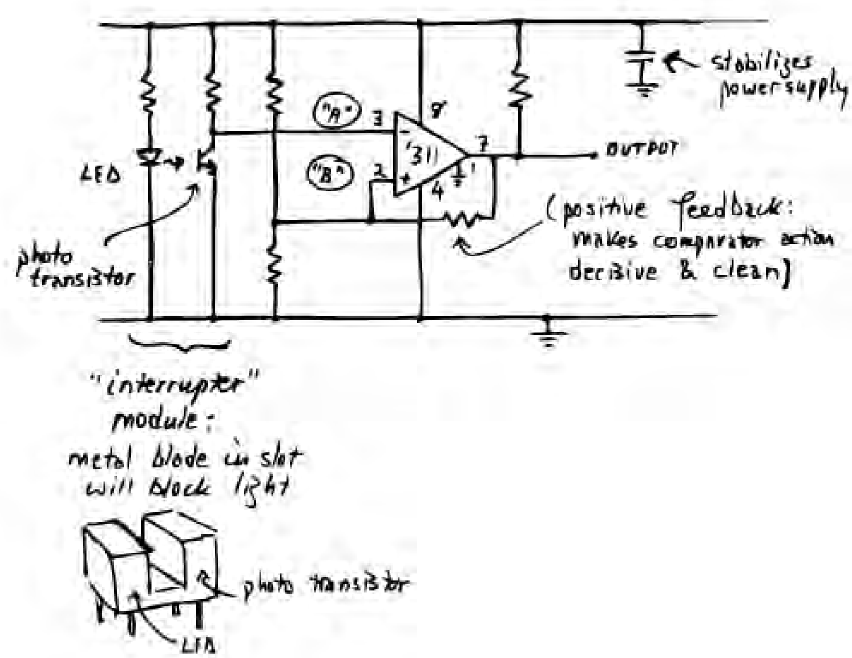

Figure 2: Timing circuit used in Lab 1

The "holes" on your breadboard are electrically connected in the following manner:

- The holes in the top two rows and the bottom two rows are connected horizontally, and the rows in each of these pairs are not connected to each other. Also, these rows are not connected between the left and right halves of the board.
- The holes in the two groups of five rows each are connected vertically but not horizontally, *i.e.* each column in these groups is an equipotential. But, the columns in these two groups are not connected to each other; each equipotential for this region of the board contains five holes.

The circuit that was put together works in the following way. The triangle on the right side of the diagram is a special kind of operational amplifier called a comparator. This device compares the two input voltages at points A and B. Depending on which voltage is greater, the comparator will output either 0 volts or 5 volts.

Your Kater's pendulum is going to swing between the LED and receiver. When the pendulum is not between the two, the receiver sees light, and the output of the circuit is 5 volts. When the pendulum is between the two circuit elements, no light reaches the receiver and the output is 0 volts. A frequency counter uses the output of this circuit to measure the period of the pendulum.

3.4 Theory-Error Analysis

In this experiment, we are going to "calculate" g by measuring two quantities, the period T and the length L of the pendulum. There are uncertainties in these measurements and thus there will be an uncertainty associated with our final value of g. To determine this uncertainty, it is necessary to use a technique known as "error analysis", or, more specific to what we will be doing, "error propagation". The following summary and formulas are all you will need to know for this lab, but a good reference on this subject is John Taylor, *An Introduction to Error Analysis.*

- If δq is the uncertainty on a measured quantity q, then $\frac{\delta q}{q}$ is the fractional uncertainty on that quantity.
- If q is a measured quantity times a constant, say $q = Bx$, then the error on q is $\delta q = |B|\delta x$.
- If q is the sum of measured quantities, $q = x + w + t$, then the error on q depends on the errors on each of the quantities in the following way:

$$\delta q = \sqrt{(\delta x)^2 + (\delta w)^2 + (\delta t)^2}.$$

- If q is a product of measured quantities, say $q = \frac{xy}{uv}$, then the fractional error on q is

$$\frac{\delta q}{q} = \sqrt{(\frac{\delta x}{x})^2 + (\frac{\delta y}{y})^2 + (\frac{\delta u}{u})^2 + (\frac{\delta v}{v})^2}.$$

- It is also true that if $q = x^n$, then $\frac{\delta q}{|q|} = |n|\frac{\delta x}{x}$.
- The most general equation states that if q is a function of independent variables x,y,z,... or $q = f(x, y, z, ...)$, then the error on q depends on the errors on each of the quantities in the following way:

$$\delta q = \sqrt{(\frac{\partial q}{\partial x})^2(\delta x)^2 + (\frac{\partial q}{\partial y})^2(\delta y)^2 + (\frac{\partial q}{\partial z})^2(\delta z)^2 + ...}$$

You will find at the end of these notes a handwritten discussion (by our former lab TF's Ehab and Daniel) treating error propagation, and considering among other issues, what happens if variables are correlated.

3.5 Experiment

Procedure

We will carry out our measurement of g using these steps:

1. Test the circuit using the card sensor. Check that the infrared LED (pretty well hidden in the black plastic "interrupter" housing) is emitting light: place an IR sensor "card" into the slot. Then send the output of the circuit to the frequency/period counter and test that the counter actually 'counts'. You will want to set it to measure period rather than frequency.
2. Set-up your equipment so that the pendulum swings cleanly through your LED/receiver configuration.

- Use a scope: It is a good idea to use an oscilloscope to watch the interrupter circuit's output: the 'scope will tell you whether the electronics is working about right. A digital scope can show you, in addition, whether the waveform's edge is "clean:" showing a single transition.
- Use a *digital* scope: a digital scope will give a good image of a very slow waveform, since the trace persists as long as you like on the screen, whereas an analog scope's trace fades rapidly. We can help you with the scope's knob-twiddling: a scope is a complicated instrument!

3. The counter/period-meter settings can be quite fussy–and, wrongly set, can produce bad data. The "sensitivity" setting is especially critical: when set too low, it fails to trigger the counter, and you get no new readings at all; that's not so bad, because you'll know it's not working. More dangerous is the fact that when sensitivity is set too high, the counter may trigger on both rising and falling edges of the waveform–giving a time that is much less than the true period.

4. Less critical is the "delay" or "sample rate" setting on the counter: adjust this so that the instrument updates only occasionally, so as to give yourself time to write the period values that you see. A new reading is indicated by either a flash of an LED (on one type of counter) or a flash of the character "C" (on the other type of counter).

5. To determine that the Kater's pendulum has been set properly, measure the period about each knife edge, and adjust the weights until the two periods are identical, to within about 1% of each other.

6. Record twenty-five periods. Find $T_{average}$, σ_T, and δT ($\delta T = \frac{\sigma_T}{\sqrt{N}}$; this will be discussed in your lab section). Measure L (using the iron bar) and estimate the reading error δL. Determine g using $T_{average}$, and find the uncertainty in g.

Conclusions and Questions

Does your range for g include the established value, $g \approx 9.80665$ m s^{-2}? What is the mathematical definition of and the intuitive meaning of σ? What are the sources of error? What limits the accuracy of the measurement?

lb1_jan07c.tex; note on error correlation follows; January 31, 2007

Lab 1: Remarks on Error Analysis (2/99)

(1) Uncorrelated Errors in Error Propagation

The formulas for error propagation on the Error Analysis section of the lab are special cases of a more general result:

Given a function of n uncorrelated variables,

$$f = f(x_1, \ldots, x_n),$$

the estimated error δf on f as a result of errors in the measurement of x_i is given by

$$\delta f = \sqrt{\sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 \delta x_i^2},$$

where δx_i is the estimated error on the measurement of x_i.

It is crucial that $x_1, \ldots, x_n$ be uncorrelated. Here is why: Suppose $f = x^2$. You could try to propagate errors using the product formula:

$$f = x \cdot x \quad \Rightarrow \quad \frac{\delta f}{f} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta x}{x}\right)^2} = \sqrt{2}\,\frac{\delta x}{x}$$

however, x is perfectly correlated with itself!

Instead, use the general formula above:

$$f = x^2 \quad \Rightarrow \quad \frac{\partial f}{\partial x} = 2x = \frac{2x^2}{x} = \frac{2f}{x} \quad \Rightarrow \quad \delta f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \delta x^2} = \left|\frac{\partial f}{\partial x}\right| \delta x$$

$$\Rightarrow \quad \delta f = \frac{2f}{x}\,\delta x \quad \Rightarrow \quad \boxed{\frac{\delta f}{f} = 2\,\frac{\delta x}{x}}$$

In other words, the formulas for sums and products given in the handouts only work when all variables involved are uncorrelated; else, you need to re-express your function f in terms of uncorrelated variables and use the general result.

Figure 3:

(2) The Error on the Mean of Independent Measurements.

Two measurements are independent if the outcome of one does not affect the outcome of the other. Thus the errors on two such measurements are uncorrelated. Let $x_1, \ldots, x_N$ be N independent measurements of some quantity x. Then, the mean of the measurements is a function of $x_1, \ldots, x_n$

$$\langle x \rangle = \langle x \rangle (x_1, \ldots, x_N) = \frac{1}{N} \sum_{n=1}^{N} x_n$$

and the error on the mean is

$$\delta\langle x \rangle = \sqrt{\sum_{j=1,\ldots,N} \left(\frac{\partial \langle x \rangle}{\partial x_j}\right)^2 \delta x_j^2} = \sqrt{\sum_{j=1,\ldots,N} \frac{1}{N^2} \delta x_j^2} = \frac{1}{N}\sqrt{N \delta x^2} = \frac{\delta x}{\sqrt{N}},$$

where δx is the expected error on one measurement (which is the same for all measurements). This error (δx) is found from looking at the spread in measured values.

This is the reason why you make many measurements of the same quantity: by taking their average, you end up knowing the value to within $\pm \frac{\delta x}{\sqrt{N}}$.

Figure 4:
end whole of Lab 1