Lab 3: The Coupled Pendulum and a bit on the Chaotic Double Pendulum

Due Friday, March 2, 2007, before 12 Noon in front of SC 301

REV0: February 21, 2007

1 Introduction

This lab looks at coupled harmonic oscillators (one of the oldest systems of interest in physics), the double pendulum and the chaotic double pendulum (a system from one of the newest fields of physics). The lab requires the use of a digital oscilloscope, in order to detect the *frequency spectrum* of the coupled pendula. The "'scope" (normmall referred to as just a "scope"), which presents its options under menus, is relatively easy to use; but getting accustomed to its controls may take you a half hour. This encounter with a contemporary scope is of some use, in itself: not only will you use this scope again, in this course, but you are likely to meet such scopes whenever you begin to work in a lab. Digital scopes, with their ability to store waveforms, and to offer computations concerning these waveforms, are fast driving out the older analog scopes.

In preparation for the lab you may want to look at any of several treatments of coupled pendula and normal modes: for example, Chapter 3 in Georgi or in Pain or (very helpful on the particular case:) pp. 32-36 in F. Crawford, Waves (Berkeley Physics Course, volume 3 (1968)). We will try to review the basic properties of the coupled harmonic oscillator at the start of the lab session.

Reminder: "f" and ω: while angular frequency, ω, is convenient in the expressions describing the behavior of harmonic oscillators, what you will observe is not angular frequency, but *oscillations per second* or *Hertz*. As you know, $\omega = 2\pi f$.

2 The Coupled Pendulum

The coupled harmonic oscillator may be the most important classical system that you will study. The basic behavior of such systems comes up over and over again in every branch of physics. (The quantum mechanical analog differs little from the classical system.) We will study the behavior of coupled pendula in detail.

2.1 Coupled Pendula, in a Nutshell

- Modes: The behavior of many systems–pendulums, vibrations on strings, musical tones, energy levels of atoms, for example–can be explained in terms of *normal modes*. For our purposes normal modes can be though of as "stable" or preferred modes (or, to use more ordinary language,"preferred patterns") of oscillation. If we start our pendulum in one of its normal modes it will stay in that mode of oscillation. The normal modes of vibration of a pendulum have frequencies associated with them. A single pendulum is capable of only one mode of oscillation; so, for small displacements it will always swing with a particular frequency. The coupled pendulum shows more complex behavior; it has two normal modes, each with its own associated frequency.
- Linearity: The "normal modes" description of behavior is useful because, for linear systems, all of the behavior of the system is simply a linear combination of the normal modes. This means, for example, that we can describe the frequency of oscillation of either one of our compound pendulums as

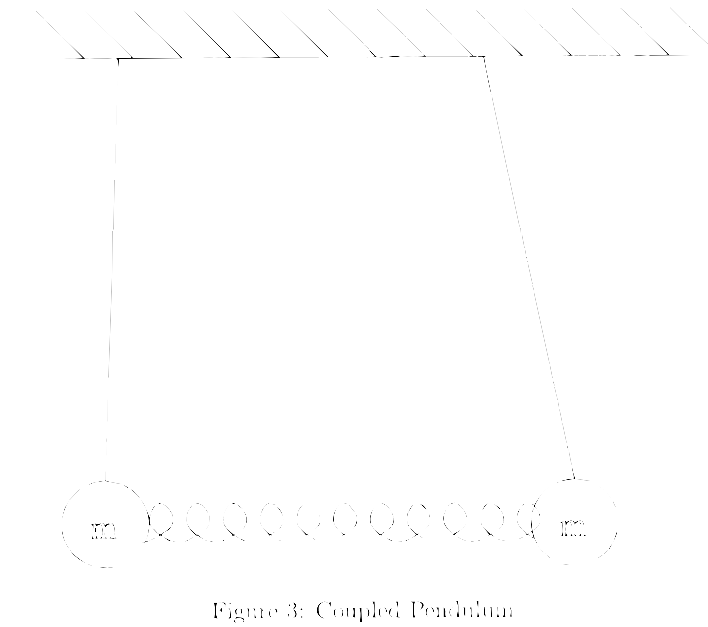


Figure 1: Coupled Pendulum

$$A \sin f_1 + B \sin f_2.$$

- Fundamental Frequencies: simple pendulum

  For a simple pendulum, force balance gives

  $$F = m\frac{d^2x}{dt^2} = -\frac{mgx}{L}$$

  and

  $$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

  The fundamental frequency ω_{simple} is equal to the square root of the coefficient of the x term or $\omega_{simple} = \sqrt{g/\text{Ł}}$

- Fundamental Frequencies: coupled pendula

  Coupled pendula exhibit two fundamental frequencies which correspond to a symmetic mode of oscillation and and asnti-symmetric mode of oscillation. **The resonant frequency of the symmetric mode is the same as that for the simple pendulum. Describe why.**

  In the anti-symmetric mode the force equation contains one more term, describing the effect of the coupling element—in our case, a spring:

  $$F = m\frac{d^2x}{dt^2} = -\frac{mgx}{L} - \frac{k}{x}$$

  and

  $$\frac{d^2x}{dt^2} + (\frac{g}{L} + \frac{k}{m})x = 0$$

This results in a modified fundamental frequency for the anti-symmetric mode, of the form

$$\omega_{anti-symmetric} = \sqrt{\frac{g}{L} + \frac{k}{m}}$$

- Superposition and ω_c and ω_e

$$\cos A + \cos B = 2\cos(\frac{A-B}{2})\cos(\frac{A+B}{2})$$

The oscillation of either one of our coupled pendulums will look like two different sine waves added together. The two superposed sine waves look like a high frequency wave that is being modulated in amplitude at a lower frequency. The higher frequency feature is called the carrier wave, ω_c, and the lower frequency modulation is called the envelope, ω_e. These two frequencies are linear combinations of the fundamental frequencies. In particular:

$$\omega_{carrier} = \frac{\omega_{symmetric} + \omega_{antisymmetric}}{2}$$

and

$$\omega_{envelope} = \frac{\omega_{symmetric} - \omega_{antisymmetric}}{2}$$

2.2 Setting up the Apparatus

Adjusting pot and scope

On each shaft on which a pendulum will be hung, a "potentiometer" is attached. A potentiometer (or "pot") is a device whose out voltage varies as the "slider" moves, making contact with a fixed resistor that runs between the two endpoint voltages. Thus, the pot produces an output voltage that varies as the shaft turns. It's a very simple device, no doubt familiar to many of you:

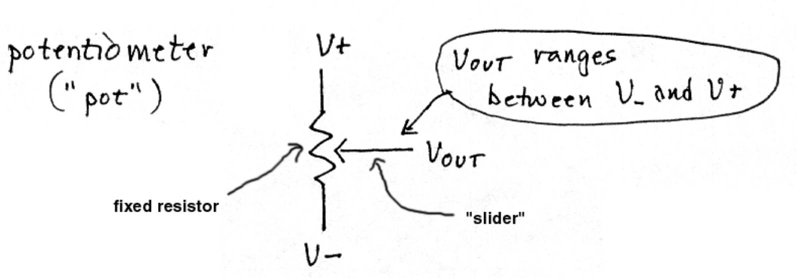


Figure 1: A potentiometer lets one pick off a "potential" between its extremes

In the lab, a pendulum is placed on the shaft of each pot[1], thus:

[1]In addition, series resistors on the ends of the pot restrict the range of voltages available at V_{OUT} to much less than the whole applied voltage range.

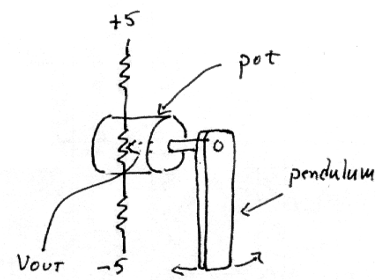


Figure 2: Pendulum hung from pot lets V_out read pendulum angle

You will drive the V_+ and V_- ends of the pot with the positive and negative outputs from a "split" power supply—one that provides both positive and negative voltages, relative to its "COMMON" terminal. Here's what the power supply front panel looks like:

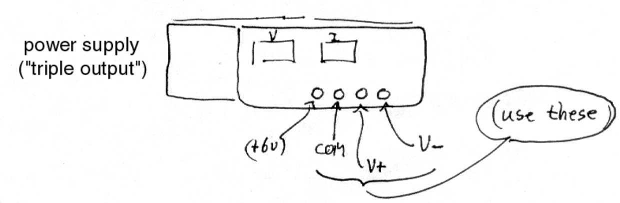


Figure 3: "Split" power supply powers the two ends of each pot

We use a *split* supply in order to place V_{OUT} close to zero volts, and to keep its swings approximately symmetric about zero.

Watch V_{OUT} with channel ONE of the digital scope. The scope *ground* (the black alligator clip) must be connected to the power supply's COMMON terminal. Now move the pendulum left and right by hand. Adjust scope *gain* so as to get a few divisions' movement of the scope trace, as the pendulum swings.

If the scope shows a voltage centered close to ground (zero volts), fine; if it does not, then loosen the clamp that holds the potentiometer in place and rotate the pot until V_{OUT} is close to zero when the pendulum is vertical.

Couple the Pendulums

When you have adjusted both pots, and are watching both pots on channels one and two of the scope, it's time to couple the two pendulums.

If the pendulums were left in the *parallel* configuration you will need to move them to the *side-by-side* configuration. Undo the bolt, move the pots so that the pendulums swing side-by-side as in figure 3 and fig. 4, below. There is a hole in the side of the aluminum beam so that you can put on the nut. Tape the middle joint of the double pendulums so that you have two simple pendulums. Then join the two pendulums with a length of spring.

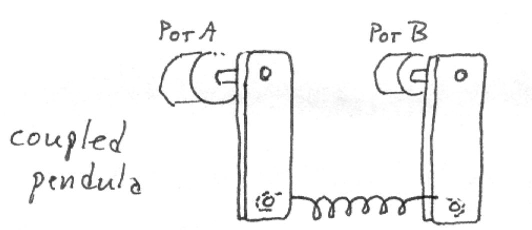


Figure 4: Two pendums coupled, again

2.3 Useful Information for Taking Data and Using the Scope

Please note that this section is not the procedure section. The exact procedure is outlined later, however you need to read and understand this section in order to take good data later.

You can then play with the *coupled* pair: if you start them swinging in unison, then they will behave like twin single pendulums—for a while, at least; if you start them with opposite phases (one to left, the other to the right) then you will see the other, quicker "mode," the oscillation whose period is affected by the spring; if you start one pendulum only, you will see it swing for a while, then gradually transfer its energy to the other...and so on.

Once you've seen this qualitative behavior, take a look at what the two pots are showing on the scope screen. One of the pots showed us, for example, the following trace:

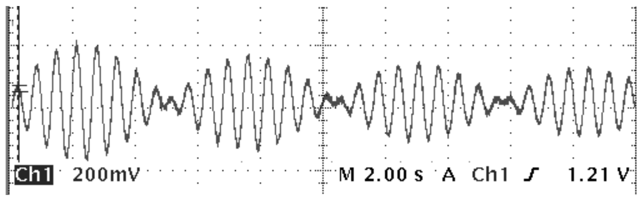


Figure 5: Scope image of one pendulum's behavior, when coupled

This image is of some value. From it, for example, you could get a good estimate of the frequency of the "envelope," at least. This envelope frequency we met earlier, as

$$\omega_{envelope} = \frac{\omega_{symmetric} - \omega_{antisymmetric}}{2}$$

But the scope can do much more to help us interpret this waveform. In particular, it can show us a *frequency spectrum* of the waveform we just looked at: it will do a "fast Fourier transform" and show us the result. Here's the FFT shown below the trace we displayed in figure 5, above. Here are the two traces: the original ("time-domain") and the FFT ("frequency-domain"):

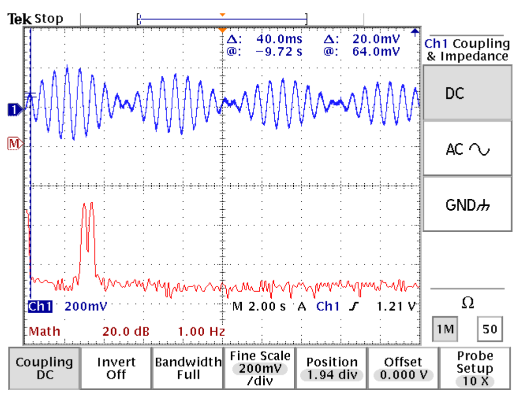


Figure 6: FFT shows frequency analysis of the original waveform

The two peaks show the frequencies of the two oscillation modes—the symmmetric (the lower frequency), and the asymmetric. We can expand the FFT to get more detailed information:

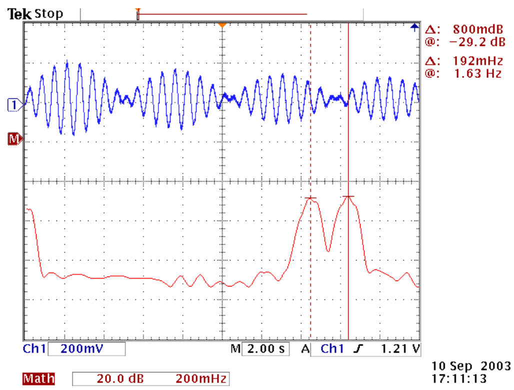


Figure 7: Expanded FFT delivers more information about the coupled oscilation

How to *expand the FFT* Step one: press the MATH button; and then increase the "sweep rate," using the knob labelled HORIZONTAL. You will notice that the region displayed narrows in frequency, and you can see that on the bar at the top of the scope display. In order to see the interesting part of the FFT, in this case very-low-frequency peaks, you must display the range of frequencies close to zero. To achieve this, use the HORIZONTAL POSITION knob to move the region that you want to watch in detail to the leftmost end of the display bar (detail below):

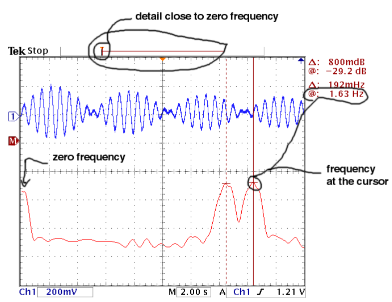
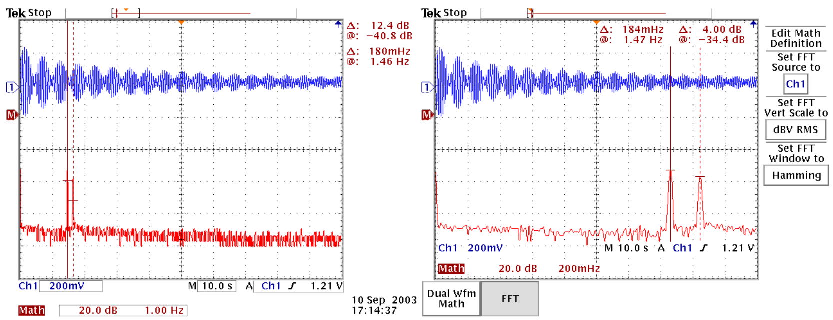


Figure 8: Use horizontal controls (sweep and position) to watch low frequencies

In the figures above (figs. 8 and 7) we have also taken advantage of the scope's vertical "cursors" to find the frequencies of the two peaks—and the difference between these two frequencies, here, 182mHz. From this difference you could predict the "envelope" frequency.

Patience is Rewarded

You will discover that you can get better resolution if you take your data more slowly. Here, we slowed the scope's "sweep rate" by a factor of five, from 2 seconds/division ("2 s/div") to 10 s/div. Gathering the data took more than 1 1/2 minutes. But the frequency peaks are better defined, as you can see.



time-domain and frequency domain: as gathered... ...and expanded to show frequency detail

Figure 9: Slower acquisition rewards you with better frequency resolution

In fact, no *new* information appears; what was dimly-visible before is now just a lot prettier.

From these screen images you should extract the frequencies of *symmetric* and *anti-symmetric* oscillation, and the difference between these two, from which you can predict *envelope* frequency. Don't forget that the cursors can help you to find all these values. Then compare your envelope prediction against what you observe, as we have said above.

Procedure: Coupled Pendulums

- Excite the 2 normal modes and find the 2 normal mode frequencies by:
  - Using the time waveform and measuring 1/T.
  - using the FFT and zooming in on the peak.

  Do not forget to estimate the uncertainty in your normal mode frequencies. First estimate the uncertainty in time from the time waveform measurements and convert this time uncertainty into frequency uncertainty. Then estimate the uncertainty in the FFT measurements by assuming that the full width at half of the maximum amplitude (FWHM) of the FFT peak is two times the uncertainty. The frequency measurements of your fundamental modes should agree (to within uncertainty) for your two methods.

- Excite a non-normal mode and do the following:
  - Set one of the pendulums in motion. Optimize the system (as is described earlier) so that you have good coupling data on your scope screen. Print out the time data and tape or staple it into your notebook.[2]
  - Use an FFT to find the 2 peaks of the normal modes together and print them out as well. Tape the FFT printout into your lab book and label the normal modes with their frequencies. Verify that they these frequencies correspond to the frequencies of the peaks found in Part 1.

[2]Here's how printing works: the scope offers a parallel connector on its back; plug into that connector the long cable that drives the printer that's on a rolling cart.On the scope, select "utility" and from the set of options apparent at the lefmost menu button, choose hardcopy. Print to CENTRONICS PORT, in format LASERJET. Start the print by pushing the "hardcopy" button, lower left of scope screen.

  - Measure the frequency of the carrier and envelope from the time waveform then verify the relationships:
    $\omega_{carrier} = \frac{\omega_{symmetric}+\omega_{antisymmetric}}{2}$ and
    $\omega_{envelope} = \frac{\omega_{symmetric}-\omega_{antisymmetric}}{2}$

- Change the magnitude of coupling by changing the spring constant K.

  Do this by putting springs in parallel and in series and measuring the normal modes and the envelope frequency. Comment on the effect on the two normal-mode frequencies. Also note and describe the effect of the spring constant upon the rate of energy transfer between the two pendulums. Check your observations against what the equations predict and comment.

3 The Compound Pendulum or Double Pendulum

3.1 The Compound Pendulum, in a Nutshell

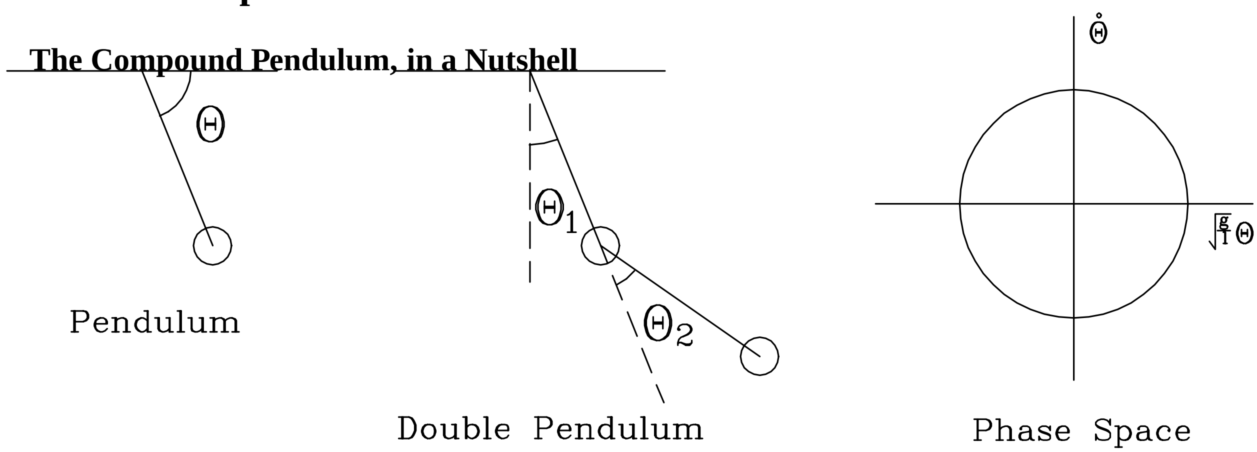


Figure 2: Pendulum, Double Pendulum and Phase Space

Consider the case of the double pendulum shown in figure 2. It turns out that for small angles we can describe the motion of the pendulum with 2 coupled equations resulting in 2 normal modes and 2 fundamental frequencies. For larger angles, however, the equations of motion become *non-linear*. These equations can be solved numerically on a computer. When this is done the behavior seems *chaotic*. What is chaotic?

A definition of chaos is that nearby trajectories in phase space tend to diverge exponentially in time.

What is phase space? Consider the simple pendulum. As we have seen for small angles $\sin\theta \approx \theta$ and

$$U = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2 \quad (1)$$

$$\rightarrow K = \dot{\theta}^2 + (g/\ell)\theta^2 \quad (2)$$

The last equation is that of a circle in $\dot{\theta},\ \sqrt{g/\ell}\,\theta$ space. The *trajectory* that the particle takes in phase space is along the circle. In the case of a damped harmonic oscillator the trajectory is a spiral. Consider a trajectory that is started with the initial condition $\theta_0, \dot{\theta}_0$ and a trajectory started with almost exactly the same initial conditions $\theta_1, \dot{\theta}_1$. These two trajectories are (at least initially) *nearby trajectories*. Now the above definition of chaos should make sense. If the distance (s) between nearby trajectories is phase space increases as an exponential function of time ($s \propto e^{\lambda t}$) then the system is chaotic. (It turns out that in order for these trajectories to really diverge exponentially, the phase space variables must be chosen properly. For our system the simple variables chosen are not the right ones, but the idea is the same.) Don't let this discussion throw you: we're really saying something very simple in somewhat high-fallutin terms!

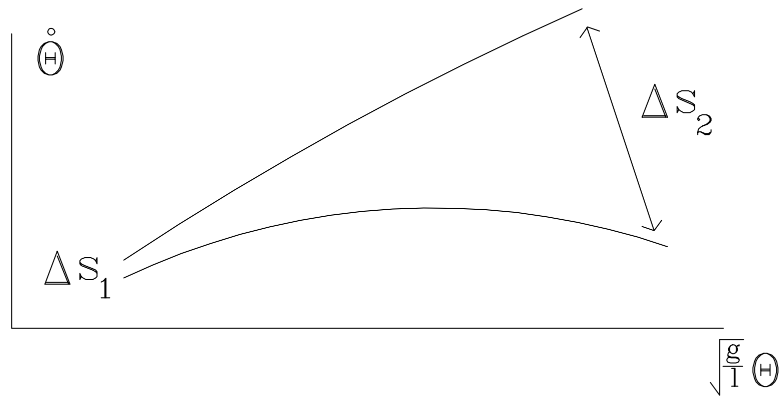


Figure 3: Divergence in Phase Space

3.2 Procedure for the Compound Pendulum

During the lab and after most of your work on the double pendulum, your lab TF will present a **very** short sketch of chaos. The double pendulums we have set up are chaotic for large angle motion. In this section of the lab we will measure angle vs. time of two "identically" prepared double pendulums (each pendulum is jointed). Qualitative aspects of chaos will be demonstrated.

- Set up the pendulums in the parallel configuration. Make sure that the pots are set up so that voltage is proportional to angle over our range of interest. Begin with just one compound pendulum.
  - Excite just one of its fundamental modes. Describe the mode and sketch it in your lab book. Measure the frequency of this mode using the FFT.
  - Excite the other fundamental mode, describe it, sketch it and measure its frequency.
  - Excite a low amplitude oscillation that involves both modes. Print out the FFT, label the two frequencies and verify that they are the same as the frequencies that were measured earlier.
- To study chaos prepare the two pendulums, as carefully as possible, with identical initial conditions. Release them at the same time. Watch the two pots with two scope channels. Print out the time data. How long did it take the behavior of the two pendula to significantly diverge? Look at the FFT and print it out. Describe how the how the FFT for the "chaotic" pendulum differs from the FFT for the small amplitude compound pendulum. Explain what the difference has to do with the concept of linearity.

(End 15c_coupled_pendo_feb07c.tex: notes on Chaos follow)