the other at various distances away from directly head-on, which will cause the balls to be deflected at various angles.

As we saw in the 1-D example in Section 5.6.1, collisions are often much easier to deal with in the CM frame. Using the same reasoning (conservation of p and E) that we used in that example, we conclude that in 2-D (or 3-D) the final speeds of two elastically colliding particles must be the same as the initial speeds. The only degree of freedom in the CM frame is the angle of the line containing the final (oppositely directed) velocities. This simplicity in the CM frame invariably provides for a cleaner solution than the lab frame yields. A good example of this is Exercise 5.81, which gives yet another way to derive the above right-angle billiard result.

5.8 Inherently inelastic processes

There is a nice class of problems where the system has inherently inelastic properties, even if it doesn't appear so at first glance. In such a problem, no matter how you try to set it up, there will be inevitable kinetic energy loss that shows up in the form of heat. Total energy is conserved, of course, since heat is simply another form of energy. But the point is that if you try to write down a bunch of $(1/2)mv^2$'s and conserve their sum, then you're going to get the wrong answer. The following example is the classic illustration of this type of problem.

Example (Sand on conveyor belt): Sand drops vertically (from a negligible height) at a rate σ kg/s onto a moving conveyor belt.

(a) What force must you apply to the belt in order to keep it moving at a constant speed v?
(b) How much kinetic energy does the sand gain per unit time?
(c) How much work do you do per unit time?
(d) How much energy is lost to heat per unit time?

Solution:

(a) Your force equals the rate of change in momentum. If we let m be the combined mass of the conveyor belt plus the sand on the belt, then

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = 0 + \sigma v, \tag{5.75}$$

where we have used the fact that v is constant.

(b) The kinetic energy gained per unit time is

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = \frac{dm}{dt}\left(\frac{v^2}{2}\right) = \frac{\sigma v^2}{2}. \tag{5.76}$$

(c) The work done by your force per unit time is

$$\frac{d(\text{Work})}{dt} = \frac{F\,dx}{dt} = Fv = \sigma v^2, \tag{5.77}$$

where we have used Eq. (5.75).

(d) If work is done at a rate σv^2, and kinetic energy is gained at a rate $\sigma v^2/2$, then the "missing" energy must be lost to heat at a rate $\sigma v^2 - \sigma v^2/2 = \sigma v^2/2$.

In this example, it turned out that exactly the same amount of energy was lost to heat as was converted into kinetic energy of the sand. There is an interesting and simple way to see why this is true. In the following explanation, we'll just deal with one particle of mass M that falls onto the conveyor belt, for simplicity.

In the lab frame, the mass gains a kinetic energy of $Mv^2/2$ by the time it finally comes to rest with respect to the belt, because the belt moves at speed v. Now look at things in the conveyor belt's reference frame. In this frame, the mass comes flying in with an initial kinetic energy of $Mv^2/2$, and then it eventually slows down and comes to rest on the belt. Therefore, all of the $Mv^2/2$ energy is converted to heat. And since the heat is the same in both frames, this is the amount of heat in the lab frame, too.

We therefore see that in the lab frame, the equality of the heat loss and the gain in kinetic energy is a consequence of the obvious fact that the belt moves at the same rate with respect to the lab (namely v) as the lab moves with respect to the belt (also v).

In the solution to the above example, we did not assume anything about the nature of the friction force between the belt and the sand. The loss of energy to heat is an unavoidable result. You might think that if the sand comes to rest on the belt very "gently" (over a long period of time), then you can avoid the heat loss. This is not the case. In that scenario, the smallness of the friction force is compensated by the fact that the force must act over a very large distance. Likewise, if the sand comes to rest on the belt very abruptly, then the largeness of the friction force is compensated by the smallness of the distance over which it acts. No matter how you set things up, the work done by the friction force is the same nonzero quantity.

In other problems such as the following one, it is fairly clear that the process is inelastic. But the challenge is to correctly use $F = dp/dt$ instead of $F = ma$, because $F = ma$ will get you into trouble due to the changing mass.

Example (Chain on a scale): An "idealized" (see the comments following this example) chain with length L and mass density σ kg/m is held such that it hangs vertically just above a scale. It is then released. What is the reading on the scale, as a function of the height of the top of the chain?

First solution: Let y be the height of the top of the chain, and let F be the desired force applied by the scale. The net force on the entire chain is $F-(\sigma L)g$, with upward taken to be positive. The momentum of the entire chain (which just comes from the moving part) is $(\sigma y)\dot{y}$. Note that this is negative, because $\dot{y}$ is negative. Equating the net force on the entire chain with the rate of change in its momentum gives

$$\begin{aligned} F-\sigma Lg &= \frac{d(\sigma y\dot{y})}{dt} \\ &= \sigma y\ddot{y}+\sigma\dot{y}^2. \end{aligned} \tag{5.78}$$

The part of the chain that is still above the scale is in free fall. Therefore, $\ddot{y}=-g$. And conservation of energy gives $\dot{y}=\sqrt{2g(L-y)}$, because the chain has fallen a distance $L-y$. Plugging these into Eq. (5.78) gives

$$\begin{aligned} F &= \sigma Lg-\sigma yg+2\sigma(L-y)g \\ &= 3\sigma(L-y)g, \end{aligned} \tag{5.79}$$

which happens to be three times the weight of the chain already on the scale. This answer for F has the expected property of equaling zero when $y=L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading suddenly drops down to the weight of the chain, namely $(\sigma L)g$.

If you used conservation of energy to do this problem and assumed that all of the lost potential energy goes into the kinetic energy of the moving part of the chain, then you would obtain a speed of infinity for the last infinitesimal part of the chain to hit the scale. This is certainly incorrect, and the reason is that there is inevitable heat loss that arises when the pieces of the chain inelastically smash into the scale.

Second solution: The normal force from the scale is responsible for doing two things. It holds up the part of the chain that already lies on the scale, and it also changes the momentum of the atoms that are suddenly brought to rest when they hit the scale. The first of these two parts of the force is simply the weight of the chain already on the scale, which is $F_{\text{weight}}=\sigma(L-y)g$.

To find the second part of the force, we need to find the change in momentum, dp, of the part of the chain that hits the scale during a given time dt. The amount of mass that hits the scale in a time dt is $dm=\sigma|dy|=\sigma|\dot{y}|\,dt=-\sigma\dot{y}\,dt$, since $\dot{y}$ is negative. This mass initially has velocity $\dot{y}$, and then it is abruptly brought to rest. Therefore, the change in its momentum is $dp=0-(dm)\dot{y}=\sigma\dot{y}^2\,dt$, which is positive. The force required to cause this change in momentum is

$$F_{dp/dt}=\frac{dp}{dt}=\sigma\dot{y}^2. \tag{5.80}$$

But as in the first solution, we have $\dot{y}=\sqrt{2g(L-y)}$. Therefore, the total force from the scale is

$$\begin{aligned} F=F_{\text{weight}}+F_{dp/dt} &= \sigma(L-y)g+2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{5.81}$$

Note that $F_{dp/dt} = 2F_{\text{weight}}$ (until the chain is completely on the scale), independent of y.

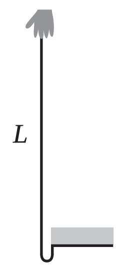

Fig. 5.16

In this example, we assumed that the chain was "ideal," in the sense that it was completely flexible, infinitesimally thin, and unstretchable. The simplest model that satisfies these criteria is a series of point masses connected by short massless strings. But in the above example, the strings actually don't even matter. You could instead start with many little unconnected point masses held in a vertical line, with the bottom one just above the scale. If you then dropped all of them simultaneously, they would successively smash into the scale in the same manner as if they were attached by little strings; the tension in the strings would all be zero. However, even though the strings aren't necessary in this chain and scale example, there are many setups involving idealized chains where they are in fact necessary, because a tension is required in them. This is evident in many of the problems and exercises for this chapter, as you will see.

An interesting fact is that even with the above definition of an ideal chain, there are some setups (in contrast with the one above) for which it is impossible to specify how the system behaves, without being given more information. This information involves the relative size of two specific length scales, as we'll see below. To illustrate this, consider the two following scenarios for the setup in Problem 5.28 (see Fig. 5.16), where a vertical ideal chain is dropped with its bottom end attached to the underneath of a support.

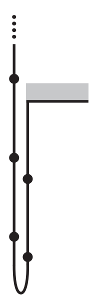

Fig. 5.17

- First scenario (energy nonconserving): Let the spacing between the point masses in our ideal chain be large compared with the horizontal span of the bend in the chain at its bottom; see Fig. 5.17. Then the system is for all practical purposes one dimensional. Each of the masses stops abruptly when it reaches the bend. This stoppage is a completely inelastic collision in the same way it was in the above example with the chain falling on the scale. Note that at any point in time, the bend consists of a massless piece of string folded back along itself (or perhaps it consists of one of the masses, if we happen to be looking at it right when a mass stops). There is no tension in this bottom piece of string (if there were, then the massless bend would have an infinite acceleration upward), so there is no tension pulling down the part of the chain on the left side of the bend. The left part of the chain is therefore in freefall.

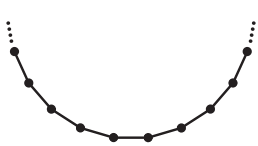

Fig. 5.18

- Second scenario (energy conserving): Let the spacing between the point masses in our ideal chain be small compared with the horizontal span of the bend in the chain at its bottom; see Fig. 5.18. The system is now inherently two dimensional, and the masses are essentially continuously distributed along the chain, as far as the bend is concerned. This has the effect of allowing each mass to gradually come to rest, so there is no abrupt inelastic stopping like there was in the first scenario. Each mass in the bend keeps the same distance from its two neighbors, whereas in the first scenario the mass that has just stopped soon sees the next mass fly directly past it before abruptly coming to rest. The process in this second scenario is elastic; no energy is lost to heat.

The basic difference between the two scenarios is whether or not there is slack in any of the strings in the bend. If there is, then the relative speed between a pair of masses changes abruptly at some point, which means that the relative kinetic energy of the masses goes into damped (perhaps very overdamped) vibrational motion in the connecting string, which then decays into the random motion of heat.[22]

If no energy is lost to heat in the second scenario, then you might think that the last infinitesimal piece of the chain will have an infinite speed. However, there isn't one *last* piece of the chain. When the left part of the chain has disappeared and we are left with only the bend and the right part, the small (but nonzero) bend is the last "piece," and it ends up swinging horizontally with a large speed. This then drags the whole chain to the side in a very visible motion (which can be traced to the horizontal force from the support), at which point we have a very noticeably two-dimensional system. The initial potential energy of the chain ends up as kinetic energy of the final wavy side-to-side motion.

A consequence of energy conservation in the second scenario is that for a given height fallen, the left part of the chain will be moving faster than the left part in the first scenario. In other words, the left part in the second scenario accelerates downward faster than the freefall g. But although this result follows quickly from energy considerations, it isn't so obvious in terms of a force argument. Apparently there exists a tension at the left end of the bend in the second scenario that drags down the left part of the chain to give it an acceleration greater than g. A qualitative way of seeing why a tension exists there is the following. A tiny piece of the chain that enters the bend from the left part slows down as it gradually joins the fixed right part of the chain. There must therefore be an upward force on this tiny piece. This upward force can't occur at the bottom of the piece, because any tension there pulls *down* on it. The force must therefore occur at the top of the piece. In other words, there is a tension at this point, and so by Newton's third law this tension pulls down on the left part of the chain, thereby causing it to accelerate faster than g. One of the tasks of Problem 5.29 is to find the tensions at the two ends of the bend.

There is a simple way to demonstrate the existence of a tension that pulls on the free part of the chain. The following setup is basically the falling-chain setup without gravity, but it still has all the essential parts. Place a rope on a

22 If the strings were ideal springs with weak spring constants, then the energy would keep changing back and forth between potential energy of the springs and kinetic energy of the masses, causing the masses to bounce around and possibly run into each other. But we're assuming that the strings in our ideal chain are essentially very rigid overdamped springs.

(fairly smooth) table, in the shape of a very thin "U" so that it doubles back along itself. Then quickly yank on one of the ends, in the direction away from the bend. You will find that the other end moves backwards, in the direction *opposite* to the motion of your hand, toward the bend (at least until the bend reaches it and drags it forward). There must therefore exist a tension in the rope to drag the other end backwards. But there's no need to take my word for it – all you need is a piece of rope. This effect is essentially the same as the one (in a simplified version, since the rope here has constant density) that leads to the crack of a whip.

Note that a perfectly flexible thin *rope*, with its continuous mass distribution, does indeed behave elastically like the second scenario above. The continuous rope may be thought of as a series of point masses with infinitesimal separation, and so this separation is much smaller than the small (but finite) length of the bend. As long as the thickness of the rope is much smaller than the length of the bend, every piece of the rope slows to a stop gradually in the original falling-chain setup, or starts up from rest gradually in the preceding "U" setup. So there is no heat loss in either setup from abrupt changes in motion.

Returning to the falling-chain system with one of our ideal chains, you might think that if the bend is made *really* small, so that the system looks one-dimensional, then it should behave inelastically like the first scenario above. However, the only relevant fact is whether the bend is smaller than the spacing between the point masses in our ideal chain. The word "small" is meaningless, of course, because we are talking about the length of the bend, which is a dimensionful quantity. It makes sense only to use the word "smaller," that is, to compare one length with another. The other length here is the spacing between the masses. If the length of the bend is large compared with this spacing, then no matter what the actual length of the bend is, the system behaves elastically like the second scenario above.

So which of the two scenarios better describes a real chain? Details of an actual experiment involving a falling chain are given in Calkin and March (1989). The results show that a real chain behaves basically like the chain in the second scenario above, at least until the final part of the motion. In other words, it is energy conserving, and the left part accelerates faster than g.[23]

Having said all this, it turns out that the energy-conserving second scenario leads to complicated issues in problems (such as the numerical integration in Problem 5.29), so for all the problems and exercises in this chapter (with the exception of Problem 5.29), we'll assume that we're dealing with the inelastic first scenario.

[23] Spur-of-the-moment (but still plenty convincing) experiments were also performed by Wes Campbell in the physics laboratory of John Doyle at Harvard.